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Elliptic Schlesinger system and Painlevé VI

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Abstract
We consider an elliptic generalization of the Schlesinger system (ESS) with positions of marked points on an elliptic curve and its modular parameter as independent variables (the parameters in the moduli space of the complex structure). This system was originally discovered by Takasaki (hep-th/9711095) in the quasi-classical limit of the $SL(N)$ vertex model. Our derivation is purely classical. ESS is defined as a symplectic quotient of the space of connections of bundles of degree 1 over the elliptic curves with marked points. The ESS is a non-autonomous Hamiltonian system with pairwise commuting Hamiltonians. The system is bi-Hamiltonian with respect to the linear and introduced here quadratic Poisson brackets. The latter are the multi-colour form of the Sklyanin–Feigin–Odesski classical algebras. The ESS is the monodromy independence condition on the complex structure for the linear systems related to the flat bundle. The case of four points for a special initial data is reduced to the Painlevé VI equation in the form of the Zhukovsky–Volterra gyrostat, proposed in our previous paper.

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1. Introduction

The Schlesinger system introduced in [1] is a system of first-order differential equations for $n > 3$ matrices $\mathbf{S}^j\,(j = 1, \ldots, n)$, depending on n points $x_k \in \mathbb{CP}^1$:

$$\partial_k \mathbf{S}^j = \frac{[\mathbf{S}^k, \mathbf{S}^j]}{x_k - x_j}, \qquad k \neq j, \quad \partial_k = \partial_{x_k}, \tag{1.1}$$

$$\partial_k \mathbf{S}^k = -\sum_{j\neq k} \frac{[\mathbf{S}^k, \mathbf{S}^j]}{x_k - x_j}. \tag{1.2}$$

This system has the Hamiltonian form with respect to the linear (Lie–Poisson) brackets on $sl(N, \mathbb{C})$. The Hamiltonian

$$H_k = \sum_{j \neq k} \frac{\langle \mathbf{S}^k \mathbf{S}^j \rangle}{x_k - x_j} \qquad (\langle\,\rangle = \mathrm{tr})$$

defines the evolution with respect to the time x_k. There exists the tau function $\exp \mathcal{F}$, related to the Hamiltonians [2]

$$\partial_k \mathcal{F} = H_k.$$

The Schlesinger equations are the monodromy preserving conditions for the linear system on $\mathbb{CP}^1$:

$$\left(\partial_z + \sum_j \frac{\mathbf{S}^j}{z - x_j} \right) \Psi = 0.$$

For 2×2 matrices and four marked points, the Schlesinger system is equivalent to the Painlevé VI equation [3]. In this case, the position of three points can be fixed as $(0, 1, \infty)$ while x_4 plays the role of an independent variable. Due to $SL(2, \mathbb{C})$ gauge symmetry, we are left with the second-order differential equation for the matrix element $(1, 2)$ of $\mathbf{S}^4$ (see, for example, [4]).

Here we replace $\mathbb{CP}^1$ by an elliptic curve and define a similar system (the elliptic Schlesinger system (ESS)). In this case, in addition to the coordinates of the marked points, a new independent variable appears inevitably. It is the modular parameter of the curve, and thereby we have an additional new Hamiltonian. This system was introduced originally by Takasaki [5]. His derivation is based on the quasi-classical limit of the quantum system living on a vertex of the $SL(N, \mathbb{C})$ generalization of the XYZ model. Here we use another approach to generic monodromy preserving systems developed earlier [6]. ESS arises as a symplectic quotient of the symplectic space of connections of principle bundles of degree 1 over the elliptic curves with n marked points.

The similar systems in their integrable versions were considered earlier in [7–9]. The latter two papers deal with a slightly different system, related to bundles of degree zero. The isomonodromic deformations corresponding to bundles of degree zero were investigated in [6, 10].

Using our approach we reproduce the main properties of the rational Schlesinger system. Namely, we prove that the ESS is a Hamiltonian system, describing interacting non-autonomous Euler–Arnold tops on coadjoint orbits attributed to the marked points with pairwise commuting Hamiltonians. The ESS is the monodromy preserving condition with respect the modular parameter of the elliptic curve and positions of the marked points. Moreover, we rewrite the ESS in terms of quadratic Poisson brackets. They are a multi-colour version of the Sklyanin–Feigin–Odesski classical algebras [11, 12]. In conclusion, for the four-point case and the matrices of order 2, we derive the Painlevé VI equation in the form of the Zhukovsky–Volterra gyrostat, proposed in our previous paper [13]. It was established there that the non-autonomous $SL(2, \mathbb{C})$ Zhukovsky–Volterra gyrostat is equivalent to the elliptic form of the Painlevé VI equation [14] proposed by Painlevé 1 year later after Fuchs (see also [15]). The corresponding isomonodromy problem on an elliptic curve is discovered only recently [16]. This paper is a continuation of [13], though it can be read independently.

2. Elliptic Schlesinger system

2.1. Definition

Let $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be an elliptic curve, with the modular parameter $\tau(\mathrm{Im}\, m\tau > 0)$ and

$$D_n = (x_1, \ldots, x_n), \qquad x_j \neq x_k, \quad x_k \in \Sigma_\tau,$$

be the divisor of non-coincident points with the condition

$$\sum x_j \in (\mathbb{Z}+\tau\mathbb{Z}). \tag{2.1}$$

Consider the space $\mathcal{P}^{(1)}_{n,N}$ of n copies of the Lie coalgebra $\mathfrak{g}^* \sim sl(N,\mathbb{C})^*$, related to the points of the divisor:

$$\mathcal{P}^{(1)}_{n,N} = \oplus_{j=1}^n \mathfrak{g}_j^*, \qquad \mathfrak{g}_j^* = \left\{ \mathbf{S}^j = \sum_{\alpha\in\tilde{\mathbb{Z}}_N^{(2)}} S_\alpha^j t^\alpha \right\}, \tag{2.2}$$

where t^α is the basis (B.7).[5]

Introduce operators acting from $\mathcal{P}^{(1)}_{n,N}$ to the dual space $\oplus_{j=1}^n \mathfrak{g}_j$:

$$\mathbf{I}_{kj} : \mathfrak{g}_k^* \to \mathfrak{g}_j, \qquad S_\gamma^k \mapsto (I_{kj})_\gamma S_\gamma^j, \qquad (I_{kj})_\gamma = \varphi_\gamma(x_j - x_k), \tag{2.3}$$

$$\mathbf{J}_{jj} : \mathfrak{g}_j^* \to \mathfrak{g}_j, \qquad S_\gamma^j \mapsto J_\gamma S_\gamma^j, \qquad J_\gamma = E_2(\check{\gamma}), \tag{2.4}$$

$$\mathbf{J}_{kj} : \mathfrak{g}_k^* \to \mathfrak{g}_j, \qquad S_\gamma^k \mapsto (J_{kj})_\gamma S_\gamma^j, \qquad (J_{kj})_\gamma = f_\gamma(x_j - x_k), \tag{2.5}$$

where $\varphi_\gamma(x)$, $E_2(\check{\gamma})$ and $f_\gamma(x)$ are defined by (B.10)–(B.15).

The positions of the marked points $x_j \in D_n$, satisfying (2.1), and the modular parameter τ are local coordinates in an open cell in the moduli space $\mathcal{M}_{1,n}$ of elliptic curves with n marked points. They play the role of times.

Definition 2.1. *The elliptic Schlesinger system (ESS) is the consistent dynamical system on* $\mathcal{P}^{(1)}_{n,N}$ *with independent variables from* $\mathcal{M}_{1,n}$*:*

$$\partial_j \mathbf{S}^k = [\mathbf{I}_{kj}(\mathbf{S}^j), \mathbf{S}^k], \qquad k \neq j, \quad \partial_k = \partial_{x_k}, \tag{2.6}$$

$$\partial_k \mathbf{S}^k = -\sum_{j\neq k} [\mathbf{I}_{jk}(\mathbf{S}^j), \mathbf{S}^k], \tag{2.7}$$

$$\partial_\tau \mathbf{S}^j = \sum_{k\neq j} \frac{1}{2\pi\imath} [\mathbf{S}^j, \mathbf{J}_{kj}(\mathbf{S}^k)] + \frac{1}{4\pi\imath} [\mathbf{S}^j, \mathbf{J}_{jj}(\mathbf{S}^j)], \tag{2.8}$$

where the commutators are understood as the coadjoint action of $\mathfrak{g}_j$ *on* $\mathfrak{g}_j^*$.

The consistency of the system will be proved below.

In the basis $t^\alpha \left(\alpha \in \tilde{\mathbb{Z}}_N^{(2)}\right)$ (B.7), the ESS takes the form

$$\partial_k S_\alpha^j = \sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}} \mathbf{C}(\gamma,\alpha) S_\gamma^k S_{\alpha-\gamma}^j \varphi_\gamma(x_j - x_k) \qquad (k \neq j), \tag{2.9}$$

[5] The upper index (1) means that $\mathcal{P}^{(1)}_{n,N}$ is equipped with the linear brackets (see (2.12)). In section 3, we introduce quadratic brackets.

$$\partial_k S^k_\alpha = \sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} \mathbf{C}(\gamma,\alpha)\sum_{j\neq k} S^j_{\alpha-\gamma}S^k_\gamma\varphi_{\alpha-\gamma}(x_k-x_j), \tag{2.10}$$

$$\partial_\tau S^k = \frac{1}{2\pi\imath}\sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} \mathbf{C}(\alpha,\gamma)\left(\sum_{k\neq j} S^k_{\alpha-\gamma}S^j_\gamma f_\gamma(x_k-x_j) + S^k_\gamma S^k_{-\gamma}E_2(\check{\gamma})\right). \tag{2.11}$$

Remark 2.1. Equations (2.9), (2.10) are consistent with the restriction on positions of the marked points (2.1), i.e. $\sum_{j=1}^n \partial_j \mathbf{S}^k = 0$.

Remark 2.2. In the rational limit ($\mathrm{Im}\, m\tau \to \infty$), (2.9) and (2.10) pass to the standard Schlesinger system (1.1), (1.2) (see (A.9)).

As in the rational case, the ESS has some fundamental properties:

- The space $\mathcal{P}^{(1)}_{n,N}$ is Poisson with respect to the linear Lie–Poisson brackets on $\mathfrak{g}^*$:

$$\left\{S^j_\alpha, S^k_\beta\right\}_1 = \delta^{jk}\mathbf{C}(\alpha,\beta)S^j_{\alpha+\beta}. \tag{2.12}$$

ESS is a non-autonomous Hamiltonian system on $\mathcal{P}^{(1)}_{n,N}$:

$$\partial_k \mathbf{S}^j = \{H_k, \mathbf{S}^j,\}_1, \qquad \partial_k = \partial_{x_k}, \quad (1,\ldots,n), \tag{2.13}$$

$$\partial_\tau \mathbf{S}^j = \{H_0, \mathbf{S}^j\}_1, \tag{2.14}$$

where

$$H_k = -\sum_{j\neq k}\langle \mathbf{I}_{kj}(\mathbf{S}^k)\mathbf{S}^j)\rangle = -\sum_{j\neq k}\sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} S^k_\gamma S^j_{-\gamma}\varphi_\gamma(x_j-x_k), \tag{2.15}$$

$$H_\tau = H_0 = -\frac{1}{2\pi\imath}\left(\sum_{k\neq j}\langle \mathbf{S}^j\mathbf{J}_{kj}(\mathbf{S}^k)\rangle + \sum_j\langle \mathbf{S}^j\mathbf{J}_{jj}(\mathbf{S}^j)\rangle\right)$$
$$= -\frac{1}{2\pi\imath}\left(\sum_{k\neq j}\sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} S^j_\gamma S^k_{-\gamma}f_\gamma(x_k-x_j) + \sum_j\sum_{\gamma\in\tilde{\mathbb{Z}}^{(2)}_N} S^j_\gamma S^j_{-\gamma}E_2(\check{\gamma})\right). \tag{2.16}$$

The brackets (2.12) are degenerate. The symplectic leaves are n copies of coadjoint orbits $\mathcal{O}_j (j = 1,\ldots,n)$ of $SL(N,\mathbb{C})$. Assume that all orbits are generic, and let $c^\mu(j)$ be corresponding Casimir functions of order $\mu(\mu = 2,\ldots,N)$. The phase space of ESS is

$$\mathcal{R}^{(1)}_{n,N} \sim \mathcal{P}^{(1)}_{n,N}\big/\{c^\mu(j) = c^\mu(j)_0\} \sim \prod \mathcal{O}_j, \tag{2.17}$$

$$\dim \mathcal{R}^{(1)}_{n,N} = nN(N-1). \tag{2.18}$$

The ESS can be considered as a system of interacting non-autonomous $SL(N,\mathbb{C})$ Euler–Arnold tops, where operators (2.3)–(2.5) play the role of the inverse inertia tensors.

- The Hamiltonians satisfy the generalized Whitham equations [17]

$$\partial_j H_k - \partial_k H_j = 0 \qquad (j,k = 0,\ldots,n). \tag{2.19}$$

In other words, the flows commute and equations (2.6)–(2.8) are consistent. These conditions provide the existence of the tau function $\exp\mathcal{F}$

$$H_j = \partial_j\mathcal{F}, \qquad H_0 = \partial_\tau\mathcal{F}.$$

- ESS is the monodromy preserving condition for flat rank N and degree 1 bundles over Σ_τ with respect to deformations of its moduli.

While the first two statements can be checked directly, the last one should be considered separately. In next subsection, we prove all of them by the symplectic reduction from a trivial, though infinite Hamiltonian system.

2.2. Derivation of ESS

Here we derive the ESS starting with a bundle over the elliptic curve Σ_τ. Deformations of the complex structure of Σ_τ allow us to introduce the times and the Hamiltonians. The ESS arises as a symplectic quotient of the space of vector bundles with respect to the action of the $SL(N,\mathbb{C})$ gauge group.

2.2.1. Vector bundles of degree 1 over elliptic curves. Let E_N be a degree 1 and rank N bundle over the elliptic curve $\Sigma_{\tau_0} \sim \mathbb{C}/(\mathbb{Z}+\tau_0\mathbb{Z})$ and $\mathrm{Conn}(E_N) = \{\mathcal{A}\}$ be the space of its C^∞ connections. It is a symplectic space with the form

$$\omega^0 = \frac{1}{2}\int_\Sigma \langle \delta\mathcal{A} \wedge \delta\mathcal{A}\rangle.$$

Let $(z,\bar z)$ be the complex coordinates on Σ_{τ_0}:

$$z = x + \tau_0 y, \qquad \bar z = x + \bar\tau_0 y \qquad (0 < x, y \leqslant 1).$$

For generic degree 1 bundles, the transition matrices corresponding to the two basic cycles can be chosen as

$$\begin{aligned} &\mathcal{A}(z+1, \bar z + 1) = Q\mathcal{A}(z,\bar z)Q^{-1}, \\ &\mathcal{A}(z+\tau_0, \bar z + \bar\tau_0) = \tilde\Lambda\mathcal{A}(z,\bar z)\tilde\Lambda^{-1} + \frac{2\pi\imath}{N}\,\mathrm{d}z, \end{aligned} \tag{2.20}$$

where $\tilde\Lambda(z,\tau) = -\mathbf{e}_N\big(-z - \frac{\tau_0}{2}\big)\Lambda$ and Q, Λ are given by (B.1) and (B.2), respectively. It means that there are no moduli parameters for degree 1 bundles.

The complex structure on Σ_τ allows us to introduce the complex structure on $\mathrm{Conn}(E_N)$. Let

$$d' = \partial + A, \qquad d'' = \bar\partial + \bar A \qquad (\partial = \partial_z, \bar\partial = \partial_{\bar z})$$

be the corresponding components of the connection $\mathcal{A}$.

In addition, we fix a quasi-parabolic structure at n marked points. It means that A has simple poles at the marked points and

$$\mathrm{Res}\, A|_{z=x_j^0} = \mathbf{S}^j = g^{-1}\mathbf{S}_0^j g \in \mathcal{O}_j \subset \mathfrak{g}_j^*$$

while $\bar A$ is regular. The symplectic form acquires the additional Kirillov–Kostant terms

$$\omega^0 = \int_\Sigma \langle \delta A \wedge \delta \bar A\rangle - \sum_{j=1}^n \big\langle \mathbf{S}_0^j g_j^{-1}\delta g_j g_j^{-1} \wedge \delta g_j\big\rangle, \qquad g_j \in SL(N,\mathbb{C}). \tag{2.21}$$

We denote the set $\mathrm{Conn}(E_N)$ with the quasi-parabolic structure at the marked points as $\tilde{\mathcal{R}}^{(1)}_{N,\tau,n}\big(\mathbf{S}_0^j\big)$.

In fact, we will work with the larger space

$$\tilde{\mathcal{P}}^{(1)}_{n,N} = \big\{\mathrm{Conn}(E_N); \oplus_{j=1}^n \mathfrak{g}_j^*\big\} = \{(A,\bar A), \mathbf{S}^j, (j = 1,\ldots,n)\}$$

equipped with the Poisson brackets

$$\{A_\alpha, \bar{A}_\beta\} = \delta_{\alpha,-\beta}, \tag{2.22}$$

$$\left\{S^j_\alpha, S^k_\beta\right\} = \delta_{jk}\mathbf{C}(\alpha,\beta)S_{\alpha+\beta}. \tag{2.23}$$

By fixing the values of the Casimir functions, we come down to $\tilde{\mathcal{R}}^1_{N,\tau,n}(\mathbf{S}^j_0)$.

2.2.2. Introducing Hamiltonians by deformation of complex structure. Deform the complex structure as

$$\begin{cases} w = z - \epsilon(z,\bar{z}), \\ \bar{w} = \bar{z}, \end{cases} \qquad \mathrm{d}w = (1-\partial\epsilon)\,\mathrm{d}z - \bar{\partial}\epsilon\,\mathrm{d}\bar{z}. \tag{2.24}$$

The Beltrami differential

$$\mu = \frac{\bar{\partial}\epsilon(z,\bar{z})}{1-\partial\epsilon(z,\bar{z})}\left(\frac{\partial}{\partial z}\otimes d\bar{z}\right) \qquad (\bar{\partial} = \partial_{\bar{z}})$$

defines the new holomorphic structure—the deformed antiholomorphic operator annihilates dw, while the antiholomorphic structure is kept unchanged:

$$\partial_{\bar{w}} = \bar{\partial} + \mu\partial, \qquad \partial_w = \partial. \tag{2.25}$$

In addition, assume that μ vanishes at the marked points $\mu(z,\bar{z})|_{x^0_j} = 0$.

Remark 2.3. In (2.24), coordinates $(w, \bar{w})$ are not complex conjugated. They are independent coordinates on the torus T^2. This choice of coordinates allows us to restrict ourselves by holomorphic dependence on μ.

We specify the dependence of μ on the positions of the marked points in the following way. Let $\mathcal{U}'_j \supset \mathcal{U}_j$ be two vicinities of the marked point x_a such that $\mathcal{U}'_j \cap \mathcal{U}'_k = \emptyset$ for $j \neq k$. Let $\chi_j(z,\bar{z})$ be a smooth function

$$\chi_j(z,\bar{z}) = \begin{cases} 1, & z \in \mathcal{U}_j\ , \\ 0, & z \in \Sigma_g \backslash \mathcal{U}'_j. \end{cases}$$

Introduce times related to the positions of the marked points $t_j = x_j - x^0_j$. Then

$$\mu_j = t_j\mu^0_j = t_j\bar{\partial}\chi_j(z,\bar{z}). \tag{2.26}$$

The dependence of the modular parameter takes the form

$$\mu_\tau = t_\tau\mu^0_0 = \frac{t_\tau}{\tau_0 - \bar{\tau}_0}\bar{\partial}(\bar{z} - z)\left(1 - \sum_{j=1}^n \chi_j(z,\bar{z})\right), \qquad t_\tau = \tau - \tau_0. \tag{2.27}$$

The functions μ^0_j $(j = 0,\ldots,n)$ can be considered as a basis in a big cell $\mathcal{M}^0_{1,n}$ of the moduli space $\mathcal{M}_{1,n}$ and the times play the role of coordinates in this basis:

$$\mu = t_\tau\mu^0_\tau + \sum_{j=1}^n t_j\mu^0_j. \tag{2.28}$$

We deform ω^0 by means of the Beltrami differential in such a way that ω^0 acquires non-trivial Hamiltonians. Let us go to a new pair of the connection components

$$(A,\bar{A}) \rightarrow (A, \bar{A}' = \bar{A} - \mu A).$$

It changes the form of ω^0 (2.21) as

$$\omega = \omega_0 - \frac{1}{2}\int_{\Sigma_\tau} \delta\langle A^2\rangle\delta\mu. \tag{2.29}$$

Expanding μ in the basis (2.28), we obtain

$$\omega = \omega^0 - \sum_{j=0}^{n} \delta\tilde{H}_j\delta t_j, \qquad t_0 = t_\tau, \tag{2.30}$$

where

$$\tilde{H}_j = \frac{1}{2}\int_{\Sigma_\tau} \langle A^2\rangle\bar{\partial}\chi_j(z,\bar{z}) \qquad (j = 1,\ldots,n), \tag{2.31}$$

$$\tilde{H}_0 = \frac{1}{2}\int_{\Sigma_\tau} \langle A^2\rangle\bar{\partial}(\bar{z}-z)\left(1 - \sum_{j=1}^{n}\chi_j(z,\bar{z})\right). \tag{2.32}$$

The form ω is defined on $\mathcal{R}^1_N(\Sigma_\tau \backslash D_n) \times \mathcal{M}^0_{1,n}$. The brackets (2.22), (2.23) and the Hamiltonians $\tilde{H}_j$ lead to the equations of motion

$$(1)\quad \partial_j\bar{A} = A\mu^0_j, \qquad (2)\quad \partial_j A = 0, \qquad (3)\quad \partial_j g_k = 0 \qquad (\partial_j = \partial_{t_j}). \tag{2.33}$$

Evidently, these flows commute pairwise. Moreover, we have from (2.22), (2.31) and (2.32) that

$$\{\tilde{H}_j, \tilde{H}_k\} = 0. \tag{2.34}$$

Remark 2.4. It easy to see that for general non-autonomous multi-time Hamiltonian systems, as for example ESS, the commutativity of flows amounts to the quasi-classical flatness

$$\partial_j H_k - \partial_k H_j + \{H_k, H_j\} = 0.$$

If, moreover, (2.34) holds, then these conditions provide the existence of the tau function $\partial_i \exp\mathcal{F} = H_i$. In particular, the tau function exists for the flows (2.33).

2.2.3. ESS as symplectic quotient. Let $\mathcal{G} = \{f(w,\bar{w})\}$ be the group of smooth maps of the deformed curve Σ_τ to $SL(N,\mathbb{C})$ with the quasi-periodicity

$$f(w+1,\bar{w}+1) = Q^{-1}f(w,\bar{w})Q, \qquad f(w+\tau,\bar{w}+\bar{\tau}) = \tilde{\Lambda}^{-1}(w)f(w,\bar{w})\tilde{\Lambda}(w). \tag{2.35}$$

Define its action on the fields as

$$\begin{aligned} &A \to f^{-1}\partial_w f + f^{-1}Af, && \bar{A} \to f^{-1}\partial_{\bar{w}} f + f^{-1}\bar{A}f,\\ &g_j \to g_j f_j, && f_j = f(z,\bar{z})|_{z=x_j}. \end{aligned} \tag{2.36}$$

The form ω is invariant with respect to this action. Therefore, we can pass to the symplectic quotient

$$\mathcal{R}^{(1)}_{N,\tau,n}\left(\mathbf{S}^j_0\right) = \tilde{\mathcal{R}}^{(1)}_{N,\tau,n}\left(\mathbf{S}^j_0\right)//\mathcal{G}.$$

Proposition 2.1.

- *The symplectic quotient is the product of the coadjoint orbits*

$$\mathcal{R}^{(1)}_{N,\tau,n}\left(\mathbf{S}^j_0\right) \sim \times_{j=1}^n \mathcal{O}_j.$$

- *The ESS is a result of the symplectic reduction of system (2.33). Its Hamiltonians (2.15), (2.16) are reduction of (2.31), (2.32) to* $\mathcal{R}^{(1)}_{N,\tau,n}\left(\mathbf{S}^j_0\right)$.
- *There exists the tau function* $\exp\mathcal{F}$ *for the ESS*

$$\partial_j \exp\mathcal{F} = H_j.$$

Proof. The symplectic quotient is characterized by the following conditions:

(i) The moment constraints

$$F(A,\bar{A})=\sum_{j=1}^{n}\mathbf{S}_j\delta(w-x_j,\bar{w}-\bar{x}_j)-N\delta(w,\bar{w})t^0,\qquad \mathbf{S}^j=g_j^{-1}\mathbf{S}_0^j g_j, \tag{2.37}$$

where $F(A,\bar{A})=\bar{\partial}A+\partial(\mu A)+[\bar{A},A]$. Note that the last term on the rhs of (2.37) comes from (2.35) and (2.36).

(ii) The gauge fixing

$$A_{\bar{w}}=0. \tag{2.38}$$

It means that generic $A_{\bar{w}}$ can be represented as the pure gauge $f^{-1}[A_{\bar{w}}]\partial_{\bar{w}}f[A_{\bar{w}}]$. As a result, $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ is described by the Lax matrix

$$L=-\partial_w ff^{-1}+fAf^{-1},\qquad f=f[A_{\bar{w}}].$$

The Lax matrix is a solution of the equation

$$\partial_{\bar{w}}L=\sum_{j=1}^{n}\mathbf{S}^j\delta(w-x_j,\bar{w}-\bar{x}_j)-N\delta(w,\bar{w})Id$$

with the quasi-periodicity (2.20). From (A.14) and (B.16), we get

$$L(w)=-\frac{1}{N}E_1(w)T_0+\sum_{j=1}^{n}\sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}}S_\gamma^j\varphi_\gamma(w-x_j)T_\gamma. \tag{2.39}$$

Here, for convenience we have used the basis T_γ instead of t^γ. We stay only with finite degrees of freedom described by the ESS variables $\mathbf{S}^j$. Thereby, the symplectic quotient $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ coincides with the phase space of the ESS (2.17). □

The following lemma completes the essential part of the proof.

Lemma 2.1.

- *The equations of motion (2.33) on the reduced space $\mathcal{R}^{(1)}_{N,\tau,n}(\mathbf{S}_0^j)$ take the Lax form*

$$\partial_k L-\partial_w M^k+[M^k,L]=0\qquad (k=0,\ldots,n), \tag{2.40}$$

where

$$M^k=-\sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}}S_\gamma^k\varphi_\gamma(w-x_k)T_\gamma\qquad (k\neq 0), \tag{2.41}$$

$$M^0=-\frac{1}{N}\partial_\tau\ln\vartheta(w|\tau)T_0+\frac{1}{2\pi\imath}\sum_{l=1}^{n}\sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}}S_\gamma^l f_\gamma(w-x_l)T_\gamma. \tag{2.42}$$

- *(2.40) coincides with the ESS (2.9)–(2.11).*

Proof. Substituting in the equation of motion for A (2.33(2))

$$A=f^{-1}\partial f+f^{-1}Lf$$

and defining $M^k=-\partial_k ff^{-1}$, we come to (2.40). It follows from (2.33(1)) that M^k satisfies the equation $\partial_{\bar{w}}M^k=-L\mu_k^0$ with the same quasi-periodicity as L for $j\neq 0$. To define M^j we have used (B.16) and (B.17). The Lax equation with $M^j (j\neq 0)$ leads directly to (2.9). The Lax equation with M^0 follows from the heat equation (A.12) and the Calogero equation (A.19).

After the reduction, the Poisson space $\tilde{\mathcal{P}}^{(1)}_{n,N}$ passes to $\mathcal{P}^{(1)}_{n,N}$ with the brackets (2.23). It follows from (2.31), (2.32) that the Hamiltonians H_j on $\mathcal{P}^{(1)}_{n,N}$ can be read off from the expansion of $\mathrm{tr}(L^2)$ on the basis of the elliptic functions

$$\frac{1}{2}\,\mathrm{tr}(L(w))^2 = \sum_{j=1}^{n}\left(H_{2,j}E_2(w-x_j) + H_{1,j}E_1(w-x_j)\right) + H_0',$$

where $H_0 = -\frac{1}{2\pi\imath}\left(H_0' - \frac{4\eta_1}{N}\right)$ and $\sum_j H_{1,j} = 0$. Here $H_{2,j} = \frac{1}{2}\sum_\gamma S^j_\gamma S^j_{-\gamma}$ are the quadratic Casimir functions corresponding to the orbits $\mathcal{O}_j$. Using (A.20) and (A.21), one can calculate the coefficients $H_{1,j}$ and H_0. They coincide with (2.15) and (2.16). The Hamiltonians commute since their pre-images commute on $\tilde{\mathcal{P}}^{(1)}_{n,N}$. Therefore, we have proved the consistency of ESS and the existence of the tau function. □

2.2.4. Isomonodromy problem. Let $\Psi \in \Gamma$ be a section of a degree 1 vector bundle over Σ_τ. Consider the linear system

$$\begin{cases}(\partial_w + A)\Psi = 0,\\ (\partial_{\bar{w}} + \bar{A})\Psi = 0,\\ \partial_k\Psi = 0 \qquad (k = 0,\ldots,n).\end{cases} \tag{2.43}$$

The compatibility condition of the first two equations is the flatness condition of the bundle. The equations of motion (2.33) are the compatibility conditions of the last equations with the first two equations. Let γ be a closed path on Σ_τ, Ψ_γ is the corresponding transformed solution and Θ_γ is the monodromy matrix

$$\Psi_\gamma = \Psi\Theta_\gamma.$$

Then the last equation implies the independence of Θ_γ on the moduli times t_k. Therefore, the equations of motion are the monodromy preserving conditions.

Let f be the gauge transformations $\Psi \to f\Psi$ that 'kills' $A_{\bar{w}}$. Then (2.43) takes the form

$$\begin{cases}(\partial_w + L)\Psi = 0,\\ \partial_{\bar{w}}\Psi = 0,\\ (\partial_k + M^k)\Psi = 0 \qquad (k = 0,\ldots,n),\end{cases} \tag{2.44}$$

where L and M^k are given by (2.39) and (2.41), (2.42), respectively. The compatibility condition of the last equation with the first one is the ESS in the Lax form (2.40). They are the monodromy preserving conditions for the linear system of the first two equations.

3. Bi-Hamiltonian structure of ESS

3.1. Quadratic Poisson algebra

Consider a complex space of dimension nN^2. We organize it in the following way. Attribute to the marked points of the divisor $D_n n$ copies of the $GL(N,\mathbb{C})$-valued elements

$$x_j \to S^j_0 T_0 + \mathbf{S}^j = \sum_{a\in\mathbb{Z}^{(2)}_N} S^j_a T_a.$$

Add to this set a variable $S_0 \in \mathbb{C}$ and define

$$\mathcal{P}^{(2)}_{n,N} = \left\{ S_0, \left(S^j_0, \mathbf{S}^j, j = 1,\ldots,n\right) \middle| \sum_{j=1}^{n} S^j_0 = 0 \right\}.$$

Proposition 3.1. *The space* $\mathcal{P}^{(2)}_{n,N}$ *is Poisson with respect to the quadratic brackets*

$$\{S_0, S_0^j\}_2 = \{S_0^j, S_0^k\}_2 = \{S_\alpha^j, S_\alpha^k\}_2 = 0, \tag{3.1}$$

$$\{S_0, S_\alpha^k\}_2 = \sum_{\gamma\neq\alpha} \mathbf{C}(\alpha,\gamma)\left(S^k_{\alpha-\gamma}S^k_\gamma E_2(\breve{\gamma}) - \sum_{j\neq k} S^j_{-\gamma}S^k_{\alpha+\gamma}f_\gamma(x_k - x_j)\right), \tag{3.2}$$

$$\begin{aligned}\{S_\alpha^k, S_\beta^k\}_2 = {} & \mathbf{C}(\alpha,\beta)S_0S^k_{\alpha+\beta} + \sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)S^k_{\alpha-\gamma}S^k_{\beta+\gamma}\mathbf{f}_{\alpha,\beta,\gamma} \\ & + \mathbf{C}(\alpha,\beta)S_0^kS^k_{\alpha+\beta}(E_1(\breve{\alpha}+\breve{\beta}) - E_1(\breve{\alpha}) - E_1(\breve{\beta})) \\ & - \mathbf{C}(\alpha,\beta)\sum_{j\neq k}\left[S_0^kS^j_{\alpha+\beta}\varphi_{\alpha+\beta}(x_k-x_j) - S_0^jS^k_{\alpha+\beta}E_1(x_k-x_j)\right]\} \\ & - 2\sum_{j\neq k}\mathbf{C}(\gamma,\alpha-\beta)S^k_{\alpha-\gamma}S^k_{\beta+\gamma}\varphi_{\beta+\gamma}(x_k-x_j)\},\end{aligned} \tag{3.3}$$

where $\mathbf{f}_{\alpha,\beta,\gamma}$ *is defined by (B.15). For* $j \neq k$,

$$\begin{aligned}\{S_\alpha^j, S_\beta^k\}_2 = {} & \sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)S^j_{\alpha-\gamma}S^k_{\beta+\gamma}\varphi_\gamma(x_j-x_k) \\ & -\mathbf{C}(\alpha,\beta)\big(S_0^jS^k_{\alpha+\beta}\varphi_\alpha(x_j-x_k) - S_0^kS^j_{\alpha+\beta}\varphi_{-\beta}(x_k-x_j)\big),\end{aligned} \tag{3.4}$$

and

$$\{S_0^j, S_\beta^k\}_2 = \begin{cases} 2\sum_\gamma \mathbf{C}(\gamma,-\beta)S^j_{-\gamma}S^k_{\beta+\gamma}\varphi_\gamma(x_k-x_j), & j\neq k, \\ -2\sum_{m\neq k}\sum_\gamma \mathbf{C}(\gamma,-\beta)S^k_{-\gamma}S^m_{\beta+\gamma}\varphi_{\beta+\gamma}(x_k-x_m), & j = k.\end{cases} \tag{3.5}$$

The brackets are extracted from the classical exchange algebra

$$\{L_1^{\text{group}}(z), L_2^{\text{group}}(w)\}_2 = \left[r(z-w), L_1^{\text{group}}(z)\otimes L_2^{\text{group}}(w)\right], \tag{3.6}$$

where r is the classical Belavin–Drinfeld r-matrix [19]

$$r(z) = \sum_\gamma \varphi_\gamma(z)T_\gamma \otimes T_{-\gamma} \tag{3.7}$$

and L^{group} is the modified Lax operator

$$L^{\text{group}} = \left(S_0 + \sum_{j=1}^n S_0^jE_1(z-x_j)\right)T_0 + \tilde{L}_j, \qquad \tilde{L}_j = \sum_\alpha S^j_\alpha\varphi_\alpha(z-x_j)T_\alpha. \tag{3.8}$$

The Jacobi identity for $\mathcal{P}^{(2)}_{n,N}$ follows from the classical Yang–Baxter equation for $r(z)$. The Poisson algebra $\mathcal{P}^{(2)}_{n,N}$ defines the structure of the Poisson–Lie group on the product of G_j attached to the marked points x_j. The proof of lemma is given in appendix C.

Remark 3.1. For $n = 1$, we come to the classical Feigin–Odesski–Sklyanin algebras [11, 12]

$$\{S_0, S_\alpha\}_2 = \sum_{\gamma\neq\alpha}\mathbf{C}(\alpha,\gamma)S_{\alpha-\gamma}S_\gamma E_2(\breve{\gamma}), \tag{3.9}$$

$$\{S_\alpha, S_\beta\}_2 = S_0S_{\alpha+\beta}\mathbf{C}(\alpha,\beta) + \sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)S_{\alpha-\gamma}S_{\beta+\gamma}\mathbf{f}(\breve{\alpha},\breve{\beta},\breve{\gamma}). \tag{3.10}$$

3.2. Bi-Hamiltonian structure

The quadratic brackets on $\mathcal{P}^{(2)}_{n,N}$ are degenerate. The function $\det L(z)$ is the generating function for the Casimir functions $C^{\mu}(j)$ [6] (see [18]). Since it is a double periodic function, it can be expanded in the basis of elliptic functions (A.6)

$$\det L(z) = C^0 + \sum_{j}^{n} C^1(j)E_1(z-x_j) + C^2(j)E_2(z-x_j) + \cdots + C^N(j)E_N(z-x_j). \tag{3.11}$$

In particular, for the second-order matrices $N = 2$

$$C^0 = S_0^2 + 4\eta_1 \sum_{j=1}^{n} \left(S_0^j\right)^2 - \sum_{\gamma} \left(\sum_{j=1}^{n} E_2(\check{\gamma}) S_\gamma^j S_\gamma^j - 2\sum_{k\neq j} S_\gamma^j S_{-\gamma}^k f_\gamma(x_k - x_j) \right), \tag{3.12}$$

$$C^1(j) = S_0 S_j + \sum_{k\neq j} S_0^j S_0^k E_1(x_j - x_k) + \sum_{k\neq j}\sum_{\gamma} S_\gamma^j S_\gamma^k \phi_\gamma(x_j - x_k), \tag{3.13}$$

$$C^2(j) = \left(S_0^j\right)^2 + \sum_{\gamma} \left(S_\gamma^j\right)^2. \tag{3.14}$$

Due to the condition

$$\sum_{j=1}^{n} C^1(j) = 0, \tag{3.15}$$

the number of the independent Casimir functions is Nn. The generic symplectic leaf

$$\mathcal{R}^2_{n,N} \sim \mathcal{P}^{(2)}_{n,N} \big/ \{(C^{\mu}(j) = C^{\mu}(j)_{(0)}),\ \mu = 1,\ldots,N,\ j = 1,\ldots,N\}$$

has dimension

$$\dim\left(\mathcal{R}^2_{n,N}\right) = nN(N-1). \tag{3.16}$$

It coincides with the dimension of the ESS phase space $\mathcal{R}^{(1)}_{N,\tau,n}\left(\mathbf{S}_0^j\right)$ defined in terms of the linear brackets.

We can extend the linear Poisson manifold $\mathcal{P}^{(1)}_{n,N}$ (2.2) by adding the variables S_0, S_0^j. In terms of the linear brackets, they are the Casimir functions and therefore preserve the phase space $\mathcal{R}^{(1)}_{N,\tau,n}\left(\mathbf{S}_0^j\right)$ (2.17).

The form of brackets (3.2), (3.5) and the Casimir functions (3.12), (3.13) suggests the following statement.

Proposition 3.2. *In terms of the quadratic brackets, the ESS takes the form*

$$\partial_k S_\alpha^j = \tfrac{1}{2}\left\{S_0^k, S_\alpha^j\right\}_2 \qquad (j,k = 1,\ldots,n),$$
$$\partial_\tau S_\alpha^j = \tfrac{1}{2}\left\{S_0, S_\alpha^j\right\}_2.$$

We have more for the second-order matrices. The Casimir functions of the quadratic brackets serve as Hamiltonians in the representations ESS by the linear brackets

$$\partial_k S_\alpha^j = \left\{C^1(k), S_\alpha^j\right\}_1 \qquad (j,k = 1,\ldots,n),$$
$$\partial_\tau S_\alpha^j = \frac{1}{2\pi\imath}\left\{C_0, S_\alpha^j\right\}_1.$$

[6] To distinguish them from the Casimir functions of the linear algebra, we denote them by capital letters.

Therefore, for $N = 2$ the trajectories of the ESS lie on the intersection of the symplectic leaves of $\mathcal{P}^{(2)}_{n,2}$ and $\mathcal{P}^{(1)}_{n,N}$. This phenomenon is a manifestation of the compatibility of the linear and the quadratic Poisson brackets. The existence of compatible Poisson structures implies the bi-Hamiltonian structure of integrable hierarchies related to these brackets [20]. We do not touch this point here.

4. Reduction to the PVI

Consider the rank 2 case ($N = 2$) with four marked points $n = 4$. We slightly change here our notation and enumerate the marked points as x_j, $j = 0, 1, 2, 3$. Replace the basis T_α with the Pauli matrices

$$T_{(1,0)} \to \sigma_3, \qquad T_{(0,1)} \to \sigma_1, \qquad T_{(1,1)} \to \sigma_2,$$

and the basis index $\alpha = 1, 2, 3$. As initial data we put the marked points on $w = 0$ and the half-periods of

$$\Sigma_\tau x_0 = 0, \qquad x_1 = \frac{\tau}{2} = \omega_2, \qquad x_2 = \frac{1+\tau}{2} = \omega_1 + \omega_2, \qquad x_3 = \frac{1}{2} = \omega_1,$$

and assume that

$$S^j_\alpha = \delta^j_\alpha \tilde{\nu}_\alpha \qquad (j = 1, 2, 3), \tag{4.1}$$

while $S^0_\alpha = S_\alpha$ are arbitrary. Since for $N = 2\check{\gamma} \sim -\check{\gamma}$, it is not difficult to see that the Hamiltonians H_j ($j = 1, 2, 3$) (2.15) vanish for this configuration, while (2.16) assumes the form

$$H_\tau = \frac{1}{2} \sum_{\gamma=1,2,3} (S_\gamma)^2 E_2(\check{\gamma}) + S_\gamma \nu'_\gamma, \qquad \nu'_\alpha = -\tilde{\nu}_\alpha\, \mathbf{e}(-\omega_\alpha \partial_\tau \omega_\alpha) \left(\frac{\vartheta'(0)}{\vartheta(\omega_\alpha)}\right)^2.$$

Therefore, the initial data (4.1) stay unchanged and we are left with the two-dimensional phase space $\mathcal{R}^{(1)} \subset \mathcal{R}^1_{4,2}$. It is described by $\mathbf{S} = (S_1, S_2, S_3)$ with the linear $sl(2, \mathbb{C})$ brackets and the Casimir function

$$c^2 = \sum_{\gamma=1,2,3} S^2_\gamma. \tag{4.2}$$

The equations of motion on $\mathcal{R}^{(1)}$ take the form of the non-autonomous Zhukovsky–Volterra gyrostat [13].

$$\partial_\tau S_\alpha = 2\imath \epsilon_{\alpha\beta\gamma} (S_\beta S_\gamma E_2(\check{\gamma}) + \nu'_\beta S_\gamma). \tag{4.3}$$

Here $\vec{S} = (S_1, S_2, S_3)$ is the momentum vector, $\vec{J} = (E_2(\omega_2), E_2(\omega_1 + \omega_2), E_2(\omega_1))$ is the inverse inertia vector and $\vec{\nu}' = (\nu'_1, \nu'_2, \nu'_3)$ is the gyrostat momentum. This equation has the bi-Hamiltonian structure based on the generalized Sklyanin algebra [13].

It was proved in [13] that there exists a transformation that allows us to pass from the elliptic form of the Painlevé VI [14] to the non-autonomous Zhukovsky–Volterra gyrostat (4.3).

The Lax matrices can be read off from their representations for the ESS (2.39), (2.42)

$$L = -\frac{1}{2}\partial_w \ln \vartheta(w; \tau)\sigma_0 + \sum_\alpha (S_\alpha \varphi_\alpha(w) + \tilde{\nu}_\alpha \varphi_\alpha(w - \omega_\alpha))\sigma_\alpha,$$

$$M = -\frac{1}{2}\partial_\tau \ln \vartheta(w; \tau)\sigma_0 + \sum_\alpha -S_\alpha \frac{\varphi_1(w)\varphi_2(w)\varphi_3(w)}{\varphi_\alpha(w)}\sigma_\alpha + E_1(w)L',$$

where $L' = \sum_\alpha (S_\alpha \varphi_\alpha(w) + \tilde{\nu}_\alpha \varphi_\alpha(w - \omega_\alpha))\sigma_\alpha$. The former matrix defines the linear problem for (4.3) in the form (2.44).

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Appendix A. Elliptic functions

We assume that $q = \exp 2\pi\,\mathrm{i}\tau$, where τ is the modular parameter of the elliptic curve E_τ.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n\, \mathbf{e}\left(\frac{1}{2}n(n+1)\tau + nz\right) = (\mathbf{e} = \exp 2\pi\imath). \tag{A.1}$$

The Eisenstein functions:

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \qquad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{A.2}$$

where

$$\eta_1(\tau) = \frac{24}{2\pi\,\mathrm{i}}\frac{\eta'(\tau)}{\eta(\tau)}, \qquad \eta(\tau) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n) \tag{A.3}$$

are the Dedekind functions.

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log\vartheta(z|\tau), \qquad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1. \tag{A.4}$$

Relation to the Weierstrass functions:

$$\zeta(z,\tau) = E_1(z,\tau) + 2\eta_1(\tau)z, \qquad \wp(z,\tau) = E_2(z,\tau) - 2\eta_1(\tau). \tag{A.5}$$

The highest Eisenstein functions

$$E_j(z) = \frac{(-1)^j}{(j-1)!}\partial^{(j-2)}E_2(z) \qquad (j>2). \tag{A.6}$$

The next important function is

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}, \tag{A.7}$$

$$\phi(u,z) = \phi(z,u), \qquad \phi(-u,-z) = -\phi(u,z). \tag{A.8}$$

It has a pole at $z = 0$ and

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}\left(E_1^2(u) - \wp(u)\right) + \cdots, \tag{A.9}$$

$$\partial_u\phi(u,z) = \phi(u,z)(E_1(u+z) - E_1(u)), \tag{A.10}$$

$$\lim_{z\to 0} \ln \partial_u\phi(u,z) = -E_2(u). \tag{A.11}$$

Heat equation:

$$\partial_\tau\phi(u,w) - \frac{1}{2\pi\mathrm{i}}\partial_u\partial_w\phi(u,w) = 0. \tag{A.12}$$

Quasi-periodicity:

$$\vartheta(z+1) = -\vartheta(z), \qquad \vartheta(z+\tau) = -q^{-\frac{1}{2}}\,\mathrm{e}^{-2\pi \mathrm{i} z}\vartheta(z), \tag{A.13}$$

$$E_1(z+1) = E_1(z), \qquad E_1(z+\tau) = E_1(z) - 2\pi\mathrm{i}, \tag{A.14}$$

$$E_2(z+1) = E_2(z), \qquad E_2(z+\tau) = E_2(z), \tag{A.15}$$

$$\phi(u, z+1) = \phi(u, z), \qquad \phi(u, z+\tau) = \mathrm{e}^{-2\pi\imath u}\phi(u, z), \tag{A.16}$$

$$\partial_u\phi(u, z+1) = \partial_u\phi(u, z), \qquad \partial_u\phi(u, z+\tau) = \mathrm{e}^{-2\pi\imath u}\partial_u\phi(u, z) - 2\pi\imath\phi(u, z). \tag{A.17}$$

The Fay three-section formula:

$$\phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1+u_2, z_1)\phi(u_2, z_2-z_1) - \phi(u_1+u_2, z_2)\phi(u_1, z_1-z_2) = 0. \tag{A.18}$$

Particular cases of this formula are the functional equations

$$\phi(u, z)\partial_v\phi(v, z) - \phi(v, z)\partial_u\phi(u, z) = (E_2(v) - E_2(u))\phi(u+v, z), \tag{A.19}$$

$$\phi(u, z_1)\phi(-u, z_2) = \phi(u, z_2-z_1)(E_1(z_1) - E_1(z_2)) - \partial_u\phi(u, z_2-z_1), \tag{A.20}$$

$$\phi(u, z)\phi(-u, z) = E_2(z) - E_2(u). \tag{A.21}$$

Another important relation is

$$\phi(v, z-w)\phi(u_1-v, z)\phi(u_2+v, w) - \phi(u_1-u_2-v, z-w)\phi(u_2+v, z)\phi(u_1-v, w) = \phi(u_1, z)\phi(u_2, w)f(u_1, u_2, v), \tag{A.22}$$

where

$$\mathbf{f}(u_1, u_2, v) = E_1(v) - E_1(u_1-u_2-v) + E_1(u_1-v) - E_1(u_2+v). \tag{A.23}$$

One can rewrite the last function as

$$\mathbf{f}(u_1, u_2, v) = -\frac{\vartheta'(0)\vartheta(u_1)\vartheta(u_2)\vartheta(u_2-u_1+2v)}{\vartheta(u_1-v)\vartheta(u_2+v)\vartheta(u_2-u_1+v)\vartheta(v)}. \tag{A.24}$$

Using (A.2), (A.4) and (A.9), one can derive from (A.22) some important particular cases. One of them corresponding to $v = u_1$ (or $v = -u_2$) is the Fay identity (A.18). Another particular case comes from $u_1 = 0$ (or $u_2 = u$):

$$\phi(v, z-w)\phi(-v, z)\phi(u+v, w) - \phi(-u-v, z-w)\phi(u+v, z)\phi(-v, w) = \phi(u_1, z)(E_2(u+v) - E_2(v)). \tag{A.25}$$

If $u_2 \to -v$, then (A.22) in the first non-trivial order takes the form for $u_1 = \alpha$, $u_2 = \beta$

$$\phi(-\beta, z-w)E_1(w)\phi(\alpha+\beta, z) - \phi(\alpha, z-w)E_1(z)\phi(\alpha+\beta, w) = \phi(\alpha, z)\phi(\beta, w)(E_1(\alpha+\beta) - E_1(\alpha) - E_1(\beta)). \tag{A.26}$$

Appendix B. Lie algebra $sl(N,\mathbb{C})$ and elliptic functions

Introduce the notation

$$\mathbf{e}_N(z) = \exp\left(\frac{2\pi\mathrm{i}}{N}z\right)$$

and two matrices

$$Q = \mathrm{diag}(\mathbf{e}_N(1), \ldots, \mathbf{e}_N(m), \ldots, 1), \tag{B.1}$$

$$\Lambda = \delta_{j,j+1} \qquad (j = 1, \dots, N, \mathrm{mod}\ N). \tag{B.2}$$

Let

$$\mathbb{Z}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}), \qquad \tilde{\mathbb{Z}}_N^{(2)}) = \mathbb{Z}_N^{(2)} \setminus (0,0) \tag{B.3}$$

be the two-dimensional lattices of orders N^2 and $N^2 - 1$, respectively. The matrices $Q^{a_1}\Lambda^{a_2}, a = (a_1, a_2) \in \mathbb{Z}_N^{(2)}$, generate a basis in the group $GL(N, \mathbb{C})$, while $Q^{\alpha_1}\Lambda^{\alpha_2}, \alpha = (\alpha_1, \alpha_2) \in \tilde{\mathbb{Z}}_N^{(2)}$, generate a basis in the Lie algebra $sl(N, \mathbb{C})$. More exactly, we introduce the following basis in $GL(N, \mathbb{C})$. Consider the projective representation of $\mathbb{Z}_N^{(2)}$ in $GL(N, \mathbb{C})$:

$$a \to T_a = \frac{N}{2\pi \mathrm{i}} \mathbf{e}_N\left(\frac{a_1 a_2}{2}\right) Q^{a_1}\Lambda^{a_2}, \tag{B.4}$$

$$T_a T_b = \frac{N}{2\pi \mathrm{i}} \mathbf{e}_N\left(-\frac{a \times b}{2}\right) T_{a+b} \qquad (a \times b = a_1 b_2 - a_2 b_1). \tag{B.5}$$

Here $\frac{N}{2\pi \mathrm{i}} \mathbf{e}_N\left(-\frac{a\times b}{2}\right)$ is a non-trivial 2-cocycle in $H^2(\mathbb{Z}_N^{(2)}, \mathbb{Z}_{2N})$. The matrices $T_\alpha, \alpha \in \tilde{\mathbb{Z}}_N^{(2)}$, generate a basis in $sl(N, \mathbb{C})$. It follows from (B.5) that

$$[T_\alpha, T_\beta] = \mathbf{C}(\alpha, \beta) T_{\alpha+\beta}, \tag{B.6}$$

where $\mathbf{C}(\alpha, \beta) = \frac{N}{\pi} \sin \frac{\pi}{N}(\alpha \times \beta)$ are the structure constants of $sl(N, \mathbb{C})$.

For $N = 2$, the basis T_α is proportional to the basis of the Pauli matrices:

$$T_{(1,0)} = \frac{1}{\pi \imath}\sigma_3, \qquad T_{(0,1)} = \frac{1}{\pi \imath}\sigma_1, \qquad T_{(1,1)} = \frac{1}{\pi \imath}\sigma_2.$$

The Lie coalgebra $\mathfrak{g}^* = sl(N, \mathbb{C})$ has the dual basis

$$\mathfrak{g}^* = \left\{ \mathbf{S} = \sum_{\tilde{\mathbb{Z}}_N^{(2)}} S_\gamma t^\gamma \right\}, \qquad t^\gamma = \frac{2\pi \imath}{N^2} T_{-\gamma}, \qquad \langle T_\alpha t^\beta \rangle = \delta_\alpha^{-\beta}. \tag{B.7}$$

It follows from (B.6) that $\mathfrak{g}^*$ is a Poisson space with the linear brackets

$$\{S_\alpha, S_\beta\} = \mathbf{C}(\alpha, \beta) S_{\alpha+\beta}. \tag{B.8}$$

The coadjoint action in these bases takes the form

$$\mathrm{ad}^*_{T_\alpha} t^\beta = \mathbf{C}(\alpha, \beta) t^{\alpha+\beta}. \tag{B.9}$$

Let $\breve{\gamma} = \frac{\gamma_1 + \gamma_2 \tau}{N}$. Then introduce the following constants on $\tilde{\mathbb{Z}}^{(2)}$:

$$\vartheta(\breve{\gamma}) = \vartheta\left(\frac{\gamma_1 + \gamma_2\tau}{N}\right), \qquad E_1(\breve{\gamma}) = E_1\left(\frac{\gamma_1 + \gamma_2\tau}{N}\right), \qquad E_2(\breve{\gamma}) = E_2\left(\frac{\gamma_1 + \gamma_2\tau}{N}\right), \tag{B.10}$$

$$\phi_\gamma(z) = \phi(\breve{\gamma}, z), \tag{B.11}$$

$$\varphi_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\phi_\gamma(z), \tag{B.12}$$

Define the function

$$f_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\partial_u \phi(u, z)|_{u=\breve{\gamma}} = \varphi_\gamma(z)(E_1(\breve{\gamma} + z) - E_1(\breve{\gamma})). \tag{B.13}$$

It follows from (A.10) that

$$f_\gamma(z) = \varphi_\gamma(z)(E_1(\breve{\gamma} + z) - E_1(\breve{\gamma})), \tag{B.14}$$

$$\mathbf{f}_{\alpha,\beta,\gamma} = E_1(\breve{\gamma}) - E_1(\breve{\alpha} - \breve{\beta} - \breve{\gamma}) + E_1(\breve{\alpha} - \breve{\gamma}) - E_1(\breve{\beta} - \breve{\gamma}). \tag{B.15}$$

(See (A.23).)

It follows from (A.7) that

$$\varphi_\gamma(z+1) = \mathbf{e}_N(\gamma_2)\varphi_\gamma(z), \qquad \varphi_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)\varphi_\gamma(z), \tag{B.16}$$

$$f_\gamma(z+1) = \mathbf{e}_N(\gamma_2)f_\gamma(z), \qquad f_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)f_\gamma(z) - 2\pi\imath\varphi_\gamma(z). \tag{B.17}$$

The modification of (A.22) is

$$\varphi_\gamma(z-x_j)\varphi_{-\gamma}(z-x_k) = \varphi_\gamma(x_k-x_j)(E_1(z-x_k)-E_1(z-x_j)) - f_\gamma(x_k-x_j). \tag{B.18}$$

Appendix C. Proof of proposition 3.1

We prove here that in the classical exchange relations (3.6) one can get rid of the spectral parameters (z,w). The result of this procedure is the quadratic Poisson algebra $\mathcal{P}^{(2)}_{n,N}$.

In (3.6), we have two types of matrix elements $T_\alpha\otimes T_\beta$ and $T_0\otimes T_\beta$. We compare the coefficients on both sides of (3.6). They are meromorphic quasi-periodic functions on $\Sigma_\tau\times\Sigma_\tau$.

The proof is based on two statements:

• the meromorphic quasi-periodic functions on Σ_τ with fixed quasi-periods are completely determined by their residues;

• the right-hand side of (3.6) is non-singular on the diagonal $z=w$. It follows from the fact that

$$r(z-w)\sim\frac{1}{z-w}T_\alpha\otimes T_{-\alpha}$$

is adjoint invariant.

Then we compare residues of the meromorphic functions with the same quasi-periods and poles on the lhs and rhs. It gives us the algebra $\mathcal{P}^{(2)}_{n,N}$.

First consider the left-hand side of (3.6).

(A) The matrix elements $T_\alpha\otimes T_\beta$:

(A1)

$$\left\{S^j_\alpha, S^j_\beta\right\}\varphi_\alpha(z-x_j)\varphi_\beta(w-x_j).$$

(A2) $k\neq j$:

$$\left\{S^k_\alpha, S^j_\beta\right\}\varphi_\alpha(z-x_k)\varphi_\beta(w-x_j) + \left\{S^j_\alpha, S^k_\beta\right\}\varphi_\alpha(z-x_j)\varphi_\beta(w-x_k).$$

(B) The matrix elements $T_0\otimes T_\beta$:

(B1)

$$\left\{S_0, S^j_\beta\right\}\varphi_\beta(w-x_j).$$

(B2) $k\neq j$:

$$\left\{S^k_0, S^j_\beta\right\}E_1(z-x_k)\varphi_\beta(w-x_j).$$

(B3) $k=j$:

$$\left\{S^j_0, S^j_\beta\right\}E_1(z-x_j)\varphi_\beta(w-x_j).$$

Finally, consider the matrix elements $T_0\otimes T_0$.

$$\left\{S_0, S^j_0\right\}\varphi_\gamma(z-x_j), \qquad \left\{S^k_0, S^j_0\right\}\varphi_\gamma(z-x_k)\varphi_\gamma(w-x_j).$$

But the matrix elements $T_0 \otimes T_0$ are absent on the right-hand side. Thus, we come to the first two identities in (3.1). Similarly, due to the structure of the r-matrix we do not have the matrix elements $T_\alpha \otimes T_\alpha$ on the rhs. It leads to the last identity in (3.1).

Now come to the right-hand side. We use the commutation relations (B.5) in the group $GL(N,\mathbb{C})$ and choose a pair of terms in such a way that their sum is explicitly non-singular on the diagonal $z = w$.

(C) The matrix elements $T_\alpha \otimes T_\beta$:

(C1) $k \neq j$:

$$\frac{1}{2}\sum_{\gamma\neq\alpha,-\beta} C(\gamma,\alpha-\beta)\times\big(S^k_{\alpha-\gamma}S^j_{\beta+\gamma}[\varphi_\gamma(z-w)\varphi_{\alpha-\gamma}(z-x_k)\varphi_{\beta+\gamma}(w-x_j) - \varphi_{\alpha-\beta-\gamma}(z-w)\varphi_{\alpha-\gamma}(w-x_k)\varphi_{\beta+\gamma}(z-x_j)] + S^j_{\alpha-\gamma}S^k_{\beta+\gamma}[\varphi_\gamma(z-w)\varphi_{\alpha-\gamma}(z-x_j)\varphi_{\beta+\gamma}(w-x_k) - \varphi_{\alpha-\beta-\gamma}(z-w)\varphi_{\alpha-\gamma}(w-x_j)\varphi_{\beta+\gamma}(z-x_k)].$$

(C2) $k = j$:

$$\frac{1}{2}\sum_{\gamma\neq\alpha,-\beta} C(\gamma,\alpha-\beta)\times S^j_{\alpha-\gamma}S^j_{\beta+\gamma}[\varphi_\gamma(z-w)\varphi_{\alpha-\gamma}(z-x_j)\varphi_{\beta+\gamma}(w-x_j) - \varphi_{\alpha-\beta-\gamma}(z-w)\varphi_{\alpha-\gamma}(w-x_j)\varphi_{\beta+\gamma}(z-x_j)].$$

(C3) $k \neq j, \gamma = \alpha$:[7]

$$\frac{1}{2}C(\alpha,\beta)\times\big(S^j_0S^k_{\beta+\alpha}[\varphi_{-\beta}(z-w)E_1(w-x_j)\varphi_{\beta+\alpha}(z-x_k) - \varphi_\alpha(z-w)E_1(z-x_j)\varphi_{\beta+\alpha}(w-x_k)] + S^k_0S^j_{\beta+\alpha}[\varphi_{-\beta}(z-w)E_1(w-x_k)\varphi_{\beta+\alpha}(z-x_j) - \varphi_\alpha(z-w)E_1(z-x_k)\varphi_{\beta+\alpha}(w-x_j)]\big).$$

(C4) $k = j, \gamma = \alpha$:

$$\frac{1}{2}C(\alpha,\beta)S^j_0S^j_{\beta+\alpha}[\varphi_{-\beta}(z-w)E_1(w-x_j)\varphi_{\beta+\alpha}(z-x_j) - \varphi_\alpha(z-w)E_1(z-x_j)\varphi_{\beta+\alpha}(w-x_j)].$$

(C5)

$$\frac{1}{2}C(\alpha,\beta)\big(S_0S^k_{\beta+\alpha}[\varphi_{-\beta}(z-w)\varphi_{\beta+\alpha}(z-x_k) - \varphi_\alpha(z-w)\varphi_{\beta+\alpha}(w-x_k)] + S_0S^j_{\beta+\alpha}[\varphi_{-\beta}(z-w)\varphi_{\beta+\alpha}(z-x_j) - \varphi_\alpha(z-w)\varphi_{\beta+\alpha}(w-x_j)]\big).$$

(D) The matrix elements $T_0 \otimes T_\beta$:

(D1) $k \neq j$:

$$\frac{1}{2}\sum_{\gamma\neq-\beta} C(\gamma,-\beta)\times\big(S^k_{-\gamma}S^j_{\beta+\gamma}[\varphi_\gamma(z-w)\varphi_{-\gamma}(z-x_k)\varphi_{\beta+\gamma}(w-x_j) - \varphi_{-\beta-\gamma}(z-w)\varphi_{-\gamma}(w-x_k)\varphi_{\beta+\gamma}(z-x_j)] + S^j_{-\gamma}S^k_{\beta+\gamma}[\varphi_\gamma(z-w)\varphi_{-\gamma}(z-x_j)\varphi_{\beta+\gamma}(w-x_k) - \varphi_{-\beta-\gamma}(z-w)\varphi_{-\gamma}(w-x_j)\varphi_{\beta+\gamma}(z-x_k)]\big).$$

[7] The same expression we have for $\gamma = -\beta$.

(D2) $k = j$:

$$\frac{1}{2}\sum_{\gamma\neq-\beta} C(\gamma,-\beta)\times S^j_{-\gamma}S^j_{\beta+\gamma}[\varphi_\gamma(z-w)\varphi_{-\gamma}(z-x_j)\varphi_{\beta+\gamma}(w-x_j) - \varphi_{-\beta-\gamma}(z-w)\varphi_{-\gamma}(w-x_j)\varphi_{\beta+\gamma}(z-x_j)].$$

Note that in all expressions on the rhs, the second term becomes equal to the first one after changing the order of summation $\gamma \to \alpha-\beta-\gamma$. Comparing expressions with the same quasi-periods, we pass from the functions φ to ϕ and in this way use identities from appendix A.

Consider first the matrix elements $T_\alpha \otimes T_\beta$ and the term (A1). The terms (C1) and (C3) on the rhs have the same poles and quasi-periods. Comparing the residues, we obtain (3.4).

The terms of type (A2) should be compared with (C1)–(C5). Before comparing, one should transform (C2) according to (A.22), (A.23) and (C4) according to (A.26). Then (C1)–(C5) generate the rhs of (3.3).

Now consider the matrix elements $T_0 \otimes T_\beta$.

Expression (D1) is periodic with respect to z and quasi-periodic with respect w. The residue of the poles is

$$\operatorname{Res} D1_{z=x_j,w=x_k} = -S^k_{-\gamma}S^j_{\beta+\gamma}\varphi_{-\beta-\gamma}(x_j-x_k)C(\gamma,-\beta).$$

This term being compared with (B2) contributes to the first line in (3.5). To come to second line, observe that

$$\operatorname{Res} D1_{z=x_j,w=x_j} = -S^k_{-\gamma}S^j_{\beta+\gamma}\varphi_{-\gamma}(x_j-x_k)C(\gamma,-\beta).$$

Moreover, (D1) contains also a term that is regular in z and has first poles in w.

$$\operatorname{Res} D1_{w=x_k} = -\text{const. term } S^k_{-\gamma}S^j_{\beta+\gamma}\varphi_\gamma(z-x_j)\varphi_{-\gamma}(z-x_k)C(\gamma,-\beta).$$

Using (B.18) we obtain

$$\operatorname{Res} D1_{w=x_k} = S^k_{-\gamma}S^j_{\beta+\gamma}C(\gamma,-\beta)f_\gamma(x_k-x_j).$$

It should be compared with (B1). In this way, we come to the last sum in (3.2).

Finally, consider (D2). As above, we can pass from φ to ϕ. We apply (A.25) for $v=\gamma$, $u=\beta$ and then compare it with (B1). As a result, we complete the rhs of (3.2).

Thus, we have the complete balance between lhs and rhs.

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